Chapter 2

Complex Signals

A number of signal processing applications make use of complex signals. Some examples include the characterization of the Fourier transform, blood velocity estimations, and modulation of signals in telecommunications. Furthermore, a number of signal-processing concepts are easier to derive, explain and understand using complex notation. It is much easier, for example to add the phases of two complex exponentials such as $x(t) = e^{j\phi_1} e^{j\phi_2}$, than to manipulate trigonometric formula, such as $\cos(\phi_1)\cos(\phi_2)$.

We start by introducing complex signals in Section 2.1, and treating the Fourier relations in Sec. 2.2. Among all complex signals, the so-called *analytic* signals are especially useful, and these will be considered in greater detail in Section 2.3.1.

2.1 Introduction to complex signals

A complex analog signal $x(t)$ is formed by the signal pair $\{x_R(t), x_I(t)\}$, where both $x_R(t)$ and $x_I(t)$ are the ordinary real signals. The relationship between these signals is given by:

$$x(t) = x_R(t) + jx_I(t), \tag{2.1}$$

where $j = \sqrt{-1}$. A complex discrete (or digital) signal $x(n)$ is defined in a similar manner:

$$x(n) = x_R(n) + jx_I(n). \tag{2.2}$$

A complex number x can be represented by its real and imaginary parts x_R and x_I, or by its magnitude and phase a and θ, respectively. The relationship between these values is illustrated in Fig. 2.1.

Complex signals are defined both in continuous time and discrete time:

$$x(t) = a(t)\exp(j\theta_a(t)) \qquad \text{and} \qquad x(n) = a(n)\exp(j\theta(n)), \tag{2.3}$$

where

$$\begin{aligned}
a(t) &= \sqrt{x_R^2(t) + x_I^2(t)} &\quad \text{and} \quad && a(n) &= \sqrt{x_R^2(n) + x_I^2(n)} \\
\theta(t) &= \arctan\frac{x_I(t)}{x_R(t)} &\quad \text{and} \quad && \theta(n) &= \arctan\frac{x_I(n)}{x_R(n)}. \\
x_R(t) &= a(t)\cos(\theta(t)) &\quad \text{and} \quad && x_R(n) &= a(n)\cos(\theta(n)) \\
x_I(t) &= a(t)\sin(\theta(t)) &\quad \text{and} \quad && x_I(n) &= a(n)\sin(\theta(n))
\end{aligned} \tag{2.4}$$

The magnitudes $a(t)$ and $a(n)$ are also known as *envelopes* of $x(t)$ and $x(n)$, respectively.

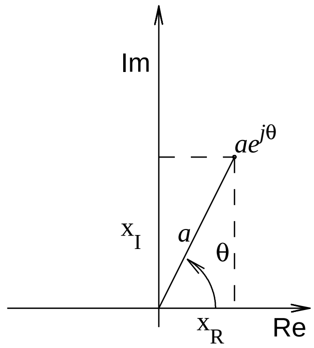


Figure 2.1: Illustration of the relationship between the real and imaginary parts of the complex number x and its magnitude and phase.

2.1.1 Useful rules and identities

Euler's formula

Many applications require to convert between a complex number and a trigonometric function. The transition is given by Euler's formula:

$$\boxed{\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ \cos\theta &= \frac{e^{j\theta}+e^{-j\theta}}{2} \\ \sin\theta &= \frac{e^{j\theta}-e^{-j\theta}}{2j} \end{aligned}} \tag{2.5}$$

Example 1 on page 59 shows how to use these identities to plot the magnitude of the spectral density function.

Useful results

Table 2.1 shows some useful identities.

r	θ	$re^{j\theta}$
1	0	$e^{j0} = 1$
1	$\pm\pi$	$e^{\pm j\pi} = -1$
1	$\pm n\pi$	$e^{\pm jn\pi} = -1$ n odd integer
1	$\pm 2\pi$	$e^{\pm j2\pi} = 1$ n
1	$\pm 2n\pi$	$e^{\pm j2n\pi} = 1$ n integer
1	$\pm\pi/2$	$e^{\pm j\pi/2} = \pm j$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \pm j$ $n = 1, 5, 6, 13, ...$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \mp j$ $n = 3, 7, 11, 15, ...$

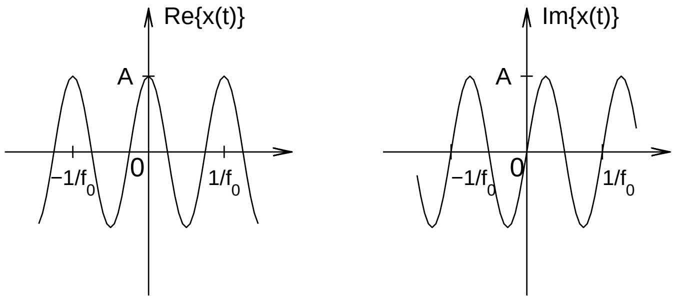
Table 2.1: Understanding some useful identities.

2.1.2 Phasors

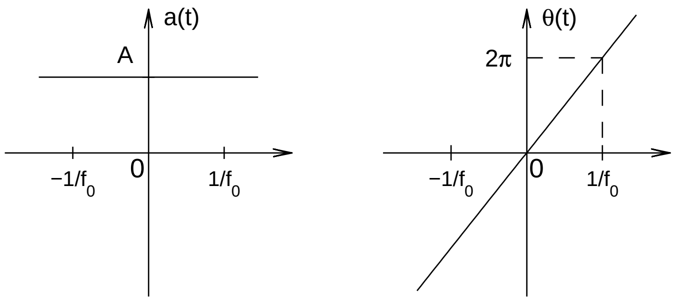
The word *phasor* is often used by mathematicians to mean any complex number. In engineering, it is frequently used to denote a complex exponential function of constant modulus and linear phase, that is a function of pure harmonic behavior. Here is an example of such a phasor:

$$x(t) = Ae^{j2\pi f_0 t}, \tag{2.6}$$

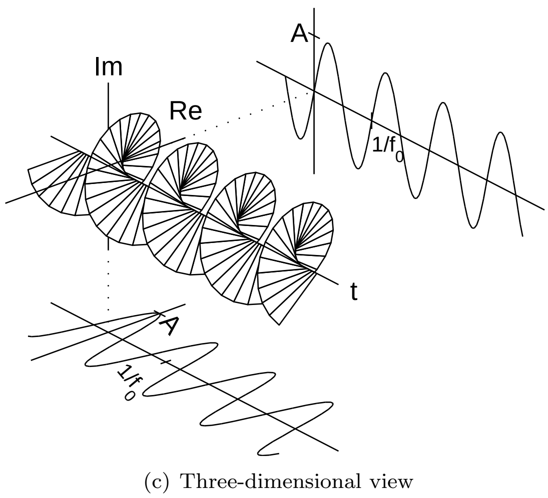
which has a constant modulus A and a linearly varying phase. It is not uncommon that the modulus and phase are plotted separately. Different ways to depict phasors are illustrated in Fig. 2.2.



(a) Real and imaginary components



(b) Modulus and phase



(c) Three-dimensional view

Figure 2.2: Different depictions of the phasor $A \exp(j2\pi f_0 t)$

Finally it must be noted that a complex valued function or phasor, whose real part is an even function and whose imaginary part is odd, is said to be hermitian. A phasor whose real part is odd and the imaginary is odd, is said to be antihermitian.

2.2 Spectrum of a complex signal

The spectrum of a complex signal can be found by using the usual expressions for the Fourier transform. In the following we will derive the spectrum $X(f)$ of the complex signal $x(t) = x_1(t) + jx_2(t)$ as a linear combination of the spectra $X_1(f)$ and $X_2(f)$ of the real-valued signals $x_1(t)$ and $x_2(t)$.

One consequence of the fact that $x(t)$ or $x(n)$ is complex, is that the typical odd/even symmetry of the spectrum are lost. It is easy to demonstrate that the following expression is valid for complex signals:

$$x^*(t) \leftrightarrow X^*(-f) \qquad \text{and} \qquad x^*(n) \leftrightarrow X^*(-f), \tag{2.7}$$

where $x^*(t)$ is the complex conjugate of $x(t)$, and $(\leftrightarrow)$ denotes a Fourier transform pair. Let the complex signal $x(t)$ be expressed in the form:

$$x(t) = x_1(t) + jx_2(t), \tag{2.8}$$

where $x_1(t)$ and $x_2(t)$ are real signals. Let their spectra be $X_1(f)$ and $X_2(f)$, respectively, i.e. $x_1(t) \leftrightarrow X_1(f)$ and $x_2(t) \leftrightarrow X_2(f)$. The real part of $x(t)$ can be expressed as[1]:

$$x_1(t) = \frac{1}{2}\left(x(t) + x^*(t)\right). \tag{2.9}$$

Using the linear property of the Fourier transform, we get:

$$G_1(f) = \frac{1}{2}\left(G(f) + G^*(-f)\right) \tag{2.10}$$

Following the same line of considerations, one gets:

$$g_2(t) = -j\frac{1}{2}\left(g(t) - g^*(t)\right) \leftrightarrow G_2(f) = -j\frac{1}{2}\left(G(f) - G^*(-f)\right). \tag{2.11}$$

If one uses the indexes R and I to denote the real and imaginary parts of a signal, the following simple relations are obtained:

$$\begin{aligned} G_R(f) &= G_{1R}(f) - G_{2I}(f) \\ G_I(f) &= G_{1I}(f) + G_{2R}(f). \end{aligned} \tag{2.12}$$

Similar relations can be derived for discrete signals too.

2.2.1 Properties of the Fourier transform for complex signals

The basic set of properties of the Fourier transform for real signals is also valid for complex signals. Table 2.2 gives a short overview of the properties of the Fourier transform for analog signals. Table 2.3 gives the equivalent properties for digital complex signals.

[1] Remember that $(a + jb)^* = a - jb$

$x(t) \leftrightarrow X(f); \qquad x_1(t) \leftrightarrow X_1(f); \qquad x_2(t) \leftrightarrow X_2(f)$
1. Linearity $$ax_1(t) + bx_2(t) \leftrightarrow aX_1(f) + bX_2(f) \tag{2.13}$$
2. Symmetry $$X(t) \leftrightarrow x(-f) \tag{2.14}$$
3. Scaling $$x(kt) \leftrightarrow \frac{1}{\lvert k \rvert} X\left(\frac{f}{k}\right) \tag{2.15}$$
4. Time reversal $$x(-t) \leftrightarrow X(-f) \tag{2.16}$$
5. Time shifting property $$x(t+t_0) \leftrightarrow X(f)e^{j2\pi f t_0}, \quad \text{where } t_0 \text{ is a real constant} \tag{2.17}$$
6. Frequency shift $$x(t)e^{-2\pi f_0 t} \leftrightarrow X(f+f_0), \quad \text{where } f_0 \text{ is a real constant} \tag{2.18}$$
7. Time and frequency differentiation $$\frac{d^p x(t)}{dt^p} \leftrightarrow (j2\pi f)^p X(f), \qquad (-j2\pi t)^p x(t) \leftrightarrow \frac{d^p X(f)}{df^p}, \quad p \text{ is a real number} \tag{2.19}$$
8. Convolution $$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f); \qquad x_1(t)x_2(t) = G_1(f) * G_2(f) \tag{2.20}$$
Parseval's theorem $$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty} \infty X_1(f)X_2^*(f)df \tag{2.21}$$

Table 2.2: Properties of the Fourier transform for complex analog signals

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$x(n) \leftrightarrow X(f); \quad x_1(n) \leftrightarrow X_1(f); \quad x_2(n) \leftrightarrow X_2(f)$	
1. Linearity $ax_1(n) + bx_2(n) \leftrightarrow aX_1(f) + bX_2(f), \qquad a \text{ and } b \text{ constants}$	(2.22)
2. The symmetry property is not relevant	
3. The scaling property is not relevant	
4. Time reversal $x(-n) \leftrightarrow X(-f)$	(2.23)
5. Time shift $x(n+n_0) \leftrightarrow X(f)e^{j2\pi f n_0 \Delta T}, \quad n_0 \text{ is an integer number}$	(2.24)
6. Frequency shift property $x(n)e^{-j2\pi f_0 n \Delta T} \leftrightarrow X(f+f_0)$	(2.25)
7. Differentiation $(j2\pi n \Delta T)^p x(n) \leftrightarrow \frac{d^p X(f)}{df^p}$	(2.26)
8. Convolution $x_1(n) * x_2(n) \leftrightarrow X_1(f)X_2(f); \quad x_1(n)x_2(n) \leftrightarrow X_1(f) * X_2(f)$	(2.27)
Parseval $\sum_{-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} X_1(f)X_2^*(f)df$	(2.28)

Table 2.3: Properties of the Fourier transform for complex digital signals.

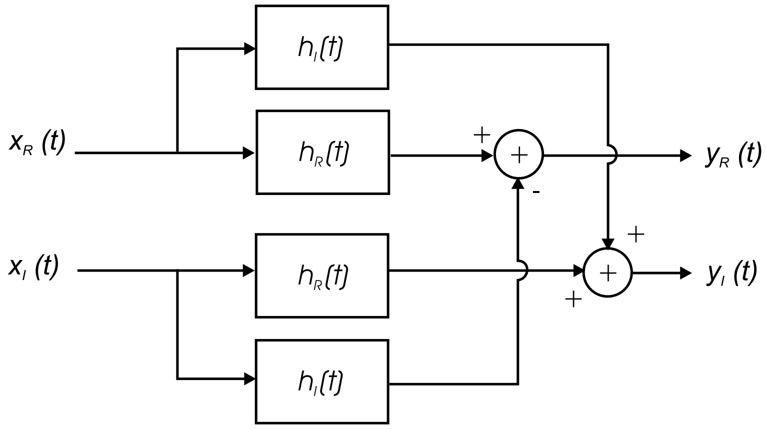

Figure 2.3: Filtration of complex signals.

2.2.2 Linear processing of complex signals

A complex signal consists of two real signals - one for the real and one for the imaginary part. The linear processing of a complex signal, such as filtration with a time-invariant linear filter, corresponds to applying the processing both to the real and the imaginary part of the signal.

The filtration with a filter, which impulse response is real, corresponds to two filtration operations - one for the real and one for the imaginary part of the signal.

Filtering a complex signal using a filter with a real-valued impulse response can be treated as two separate processes - one for the filtration of the real and one for the filtration of the imaginary component of the input signal:

$$h(t) * (a(t) + jb(t)) = h(t) * a(t) + jh(t) * b(t). \tag{2.29}$$

If the filter has a complex impulse response, then the operation corresponds to 4 real filtering operations as shown in Fig. 2.3

An example of an often-used filter with complex impulse response is the filter given by:

$$h_m(n) = \begin{cases} \frac{1}{N} e^{jm\frac{2\pi}{N}n} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{2.30}$$

The transfer function of the filter

$$H_m(f) = \frac{1}{N} \frac{\sin \pi(fN\Delta T - m)}{\sin \pi(f\Delta T - m/N)} e^{-j\pi(N-1)(f\Delta T - m/N)}, \tag{2.31}$$

is a function of the parameter m. Figure 2.4 illustrates both the impulse response and the transfer function of the filter.

2.3 Analytic signals

An analytic signal is a signal, which spectrum is “one-sided”. For analog signals this means that their spectrum is $\equiv 0$ for $f > 0$ or $f < 0$. Analytic discrete-time signals have a spectrum which is $\equiv 0$ for $-\frac{f_s}{2} < f < 0$ and in the corresponding parts of the periodic spectrum, or $0 < f < \frac{f_s}{2}$ and the corresponding parts of the periodic spectrum. The so-introduced condition for an analytic signal gives the connection between the real and the imaginary part of the complex signal.

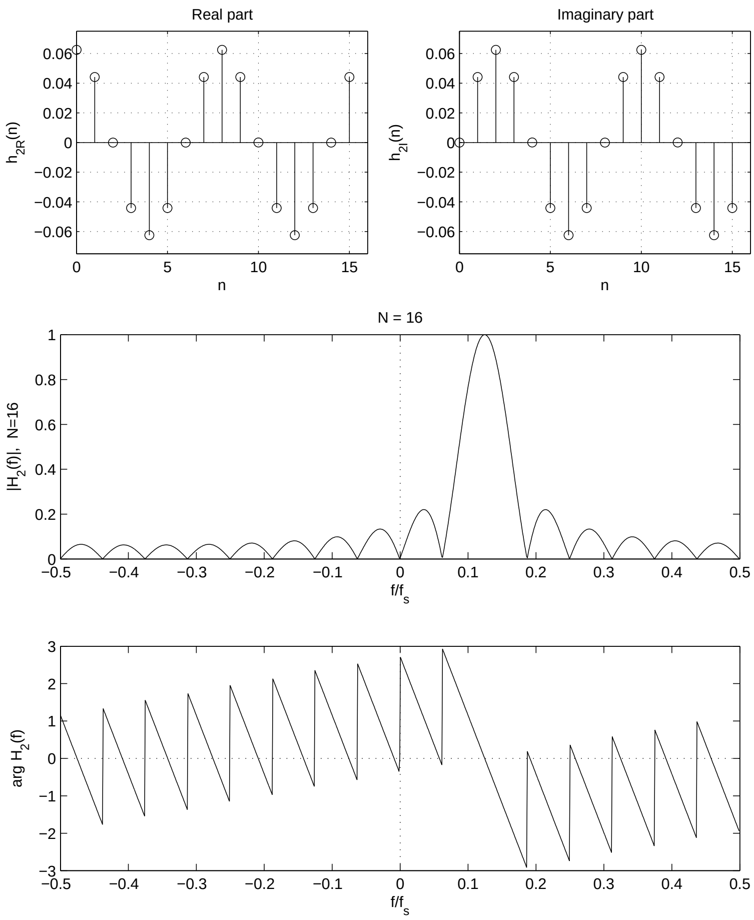

Figure 2.4: Impulse response and transfer function of a complex filter used to carry out the Discrete-time Fourier transform.

2.3.1 Analytic analog signals

If a real signal $x(t)$ with frequency spectrum X(f) is taken as a starting point, then the following relations will be valid for the respective analytic signal $z_x(t)$ and its spectrum:

$$z_x(t) \leftrightarrow Z_x(f) = \begin{cases} 2X(f) & \text{for } f > 0 \\ X(f) & \text{for } f = 0 \\ 0 & \text{for } f < 0. \end{cases} \tag{2.32}$$

This relation can be expressed in a more compact form as[2]:

$$Z_x(f) = [1 + \text{sgn}(f)]X(f). \tag{2.33}$$

Since the $\text{sgn}(f)$ is the fourier spectrum of the function $j\frac{1}{\pi t}$ ($j\frac{1}{\pi t} \leftrightarrow \text{sgn}(f)$), then the above equation is equivalent to:

$$z_x(t) = \left(\delta(t) + j\frac{1}{\pi t}\right) * x(t) \tag{2.34}$$

Here we introduce the signal $x_H(t)$, known as the Hilbert transform of $x(t)$ and given by:

$$x_H(t) = x(t) * \frac{1}{\pi t}. \tag{2.35}$$

It can be seen that

$$z_x(t) = x(t) + jx_H(t). \tag{2.36}$$

Notice that the complex conjugate $z_x^*(t)$ is also analytic with spectrum given by

$$Z_x^*(f) = \begin{cases} 0 & \text{for } f > 0 \\ X(0) & \text{for } f = 0 \\ 2X(f) & \text{for } fz < 0 \end{cases} \tag{2.37}$$

and that consequently:

$$x(t) = \frac{1}{2}\left(z_x(t) + z_x^*(t)\right). \tag{2.38}$$

If $z_x(t)$ is written in the form

$$z_x(t) = a_z(t)\exp(j\theta(t)) \tag{2.39}$$

then

$$x(t) = a_z(t)cos(\theta_z(t)) \quad \text{and} \quad x_H(t) = a_z(t)\sin(\theta(t)) \tag{2.40}$$

If the analytic signal $z_x(t)$ is filtered with a filter with a real impulse response, then the output signal $y(t)$ will be:

$$y(t) = h(t) * z_x(t) = h(t) * x(t) + jh(t) * x_H(t). \tag{2.41}$$

If the Hilbert transform of $h(t)$ is denoted by $h_H(t)$, then one gets:

$$y(t) = x(t) * (h(t) + jh_H(t)) = x(t) * z_h(t). \tag{2.42}$$

This operation is some times useful when one wants to work with analytic signals, but has only a real signal to start with. Notice that $h_H(t)$ is usually noncausal.

Using the symmetry property of the Fourier transform it can be shown that the real and imaginary parts of the spectrum of a real-signal form a Hilbert pair, that is each can be obtained from the other using a Hilbert transform. This, and a number of other properties of the Hilbert transform can be found in Table **??**. The symbol $\mathcal{H}$ is used in the table to denote the Hilbert transform, and the result is $\tilde{x}(t) = \mathcal{H}\{x(t)\}$.

[2] $\text{sgn}(f)$ returns the sign of the argument. It returns $+1$ if $f > 0$, -1 if $f < 0$, and 0 if $f = 0$.

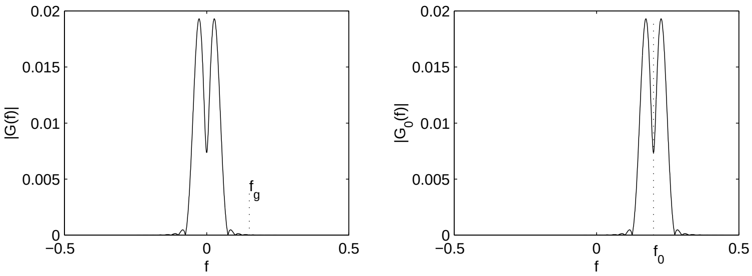

Figure 2.5: Construction of an analytic signal

2.3.2 Analytic discrete-time signals

If a real discrete-time signal $x(n) \leftrightarrow X(f)$ is used as a basis, then the corresponding analytic signal[3] $z_x(n)$ will be given as

$$z_x(n) \leftrightarrow Z_x(f) = \begin{cases} 2X(f) & \text{for } \; pf_s < f < (2p+1)\frac{f_s}{2} \\ X(f) & \text{for } \; f = pf_s \\ 0 & \text{otherwise,} \end{cases} \tag{2.43}$$

where p is an integer number. Let's consider the following spectrum:

$$Z_0(f) = \begin{cases} 1 & \text{for } \; pf_s < f < (2p+1)\frac{f_s}{2} \\ 0 & \text{for } \; f = pf_s \\ -1 & \text{for } \; (2p-1)f_s < f < pf_s. \end{cases} \tag{2.44}$$

The discrete time signal that corresponds to this spectrum is

$$j\frac{2}{n\pi}\sin^2(n\frac{\pi}{2}). \tag{2.45}$$

It follows directly that

$$z_x(t) = \left(\delta(n) + j\frac{2}{n\pi}\sin^2(n\frac{\pi}{2})\right) * g(n). \tag{2.46}$$

By analogy with the relations in Section 2.3.1 we denote the signal

$$x_H(n) = x(n) * \frac{2}{n\pi}\sin^2(n\frac{\pi}{2}) \tag{2.47}$$

as the Hilbert transform of $x(n)$. The relation between $z_x(n)$, $x(n)$ and $x_H(n)$ is given by:

$$z_x(n) = x(n) + jx_H(n). \tag{2.48}$$

Similarly to Section 2.3.1 it can be shown that

$$h(n) * z_h(n) = x(n) * z_h(n), \tag{2.49}$$

where $z_h(n) = h(n) + jh_H(n) = h(n) + jh(n) * \frac{2}{n\pi}\sin^2(n\frac{\pi}{2})$.

2.4 Instantaneous amplitude and frequency

Let's consider the band-limited real signal $x(t)$ with a band limit f_g. The amplitude spectrum of such a signal is shown in the left sub-plot of Fig. 2.5.

[3]The use of the term "analytic" in relation to discrete-time signals leads to mathematical difficulties. Many of them can, however, be circumvented if one applies the fact, that a given digital signal corresponds to an equivalent analog signal.

1. Linearity

$$\mathcal{H}\{ax_1(t) + bx_2(t)\} = a\tilde{x}_1(t) + b\tilde{x}_2(t), \quad a \text{ and } b \text{ constants} \tag{2.50}$$

2. Time shift

$$\mathcal{H}\{x(t + t_0)\} = \tilde{x}(t + t_0) \tag{2.51}$$

3. Applying two times the Hilbert transform gives the original signal

$$\mathcal{H}\{\mathcal{H}\{x(t)\}\} = -x(t) \tag{2.52}$$

4. The inverse Hilbert transform

$$x(t) = \mathcal{H}^{-1}\{x(t)\} = -\int_{-\infty}^{\infty} \frac{\tilde{x}(t)}{\pi(t-u)} du = \tilde{x}(t) * \frac{-1}{\pi t} \tag{2.53}$$

5. Even/odd property

$$\begin{aligned} x(t) \text{ even} &\Leftrightarrow \tilde{x}(t) \text{ odd} \\ x(t) \text{ odd} &\Leftrightarrow \tilde{x}(t) \text{ even} \end{aligned} \tag{2.54}$$

6. Conservation of energy

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} \tilde{x}^2(t)dt \tag{2.55}$$

7. Orthogonality

$$\int_{-\infty}^{\infty} x(t)\tilde{x}(t) = 0 \tag{2.56}$$

8. Modulation

$$\mathcal{H}\{x(t)\cos(2\pi f_0 t)\} = x(t)\sin(2\pi f_0 t) \tag{2.57}$$

if

$$X(f) = \begin{cases} X(f) & |f| \leq F, f_0 > F \\ 0 & \text{otherwise} \end{cases}$$

9. Convolution

$$\mathcal{H}\{x(t) * h(t)\} = x(t) * \tilde{h}(t) = \tilde{x}(t) * h(t) \tag{2.58}$$

10. Spectrum of a real signal

$$\mathcal{H}\{X_R(f)\} = -X_I(f), \quad \mathcal{H}\{X_I(f)\} = X_R(f) \tag{2.59}$$

where $x(t) \leftrightarrow X_R(f) + jX_I(f)$, and $x(t)$ is real and causal.

Table 2.4: Properties of the Hilbert transform.

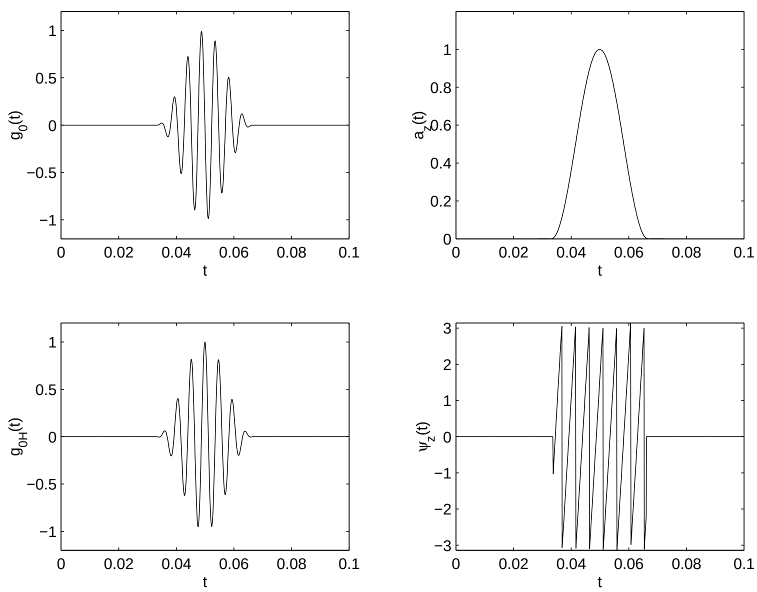

Figure 2.6: Instantaneous amplitude and phase for an analytic signal.

For this particular case an analytic signal can be created by frequency-shifting the spectrum of the signal with an offset f_0, where $|f_0| > f_g$. The frequency-shifting operation is illustrated in the right sub-plot of Fig. 2.5. In other words, we create a signal $z_0 = x(t)e^{j2\pi f_0 t}$, which is analytic if $|f_0| > f_g$. It can be represented with a real and imaginary part as:

$$z_0(t) = x_0(t) + jx_{0H}(t), \tag{2.60}$$

where $x_{0H}(t)$ is the Hilbert transform of the real signal $x_0(t)$. Using the Euler's relations, we find that:

$$x_0(t) = x(t)\cos(2\pi f_0 t), \quad \text{and} \quad x_{0H} = x(t)\sin(2\pi f_0 t). \tag{2.61}$$

If we assume that $x(t)$ gets only non-negative values ($x(t) \geq 0$), then it follows immediately that:

$$a_z(t) = x(t) \quad \text{and} \quad \theta_z(t) = 2\pi f_0 t. \tag{2.62}$$

The *instantaneous* amplitude of $z_0(t)$ is then $x(t)$, and the value $\theta_z'(t)/2\pi$ is said to be the *instantaneous* frequency of the signal. An example is given in Fig. 2.6.

In the case that $x(t)$ can both be positive *and* negative, the amplitude $a_z(t)$ will be equal to the absolute value of $x(t)$, $a_z(t) = |x(t)|$. The phase $\theta_z(t)$ of the analytic signal will jump from with $\pm\pi$ in those time instances, when $x(t)$ changes sign. The first derivative with respect to time $\theta'(t)$ will still be proportional to the instantaneous frequency, except for the points of discontinuity as shown in Fig. 2.7.

The concepts of instantaneous amplitude and frequency can be transferred formally to the typical case, in which the analytic signal $z(t)$ is given in the form $a_z(t)\exp(j\theta(t))$. Figure 2.8 shows an example in which $\theta(t)$ varies non-linearly in time.

These concepts are also applied to non-analytic signals, and the same values can be defined for discrete-time signals. The phase of the signal is given by:

$$\theta(t) = \arctan\frac{x_H(t)}{x(t)}. \tag{2.63}$$

The instantaneous frequency is proportional to the derivative of the phase with respect to time, and can be found from:

$$\frac{\theta'(t)}{2\pi} = \frac{1}{2\pi}\cdot\frac{x(t)x_H'(t) - x_H(t)x'(t)}{x^2(t) + x_H^2(t)} \tag{2.64}$$

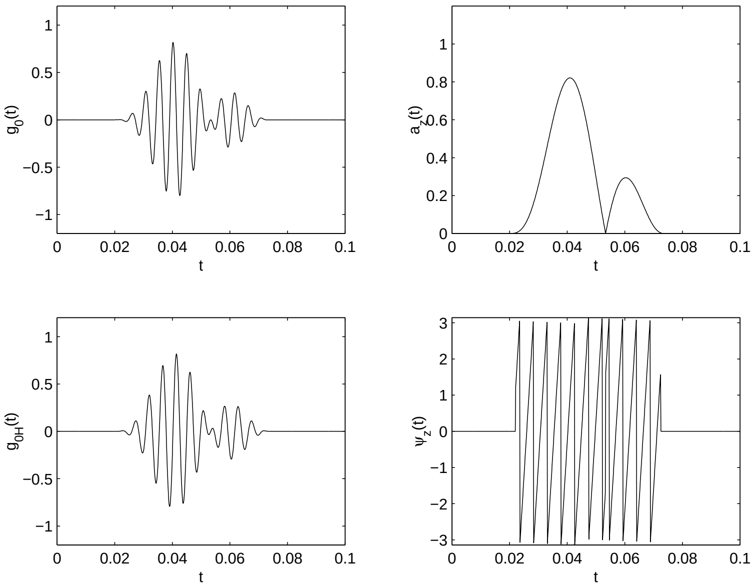

Figure 2.7: Instantaneous amplitude and phase for an analytic signal with a point of discontinuity.

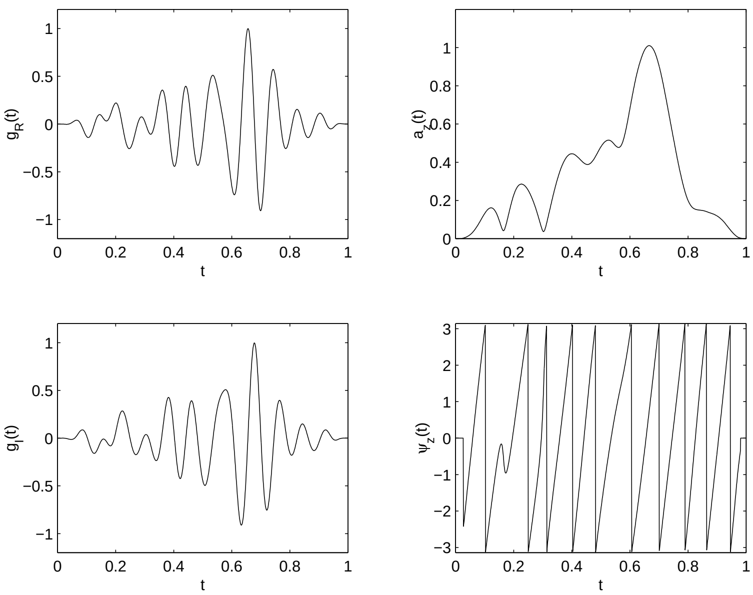

Figure 2.8: Instantaneous amplitude and phase of a signal with non-linear variation of $\theta_z(t)$.

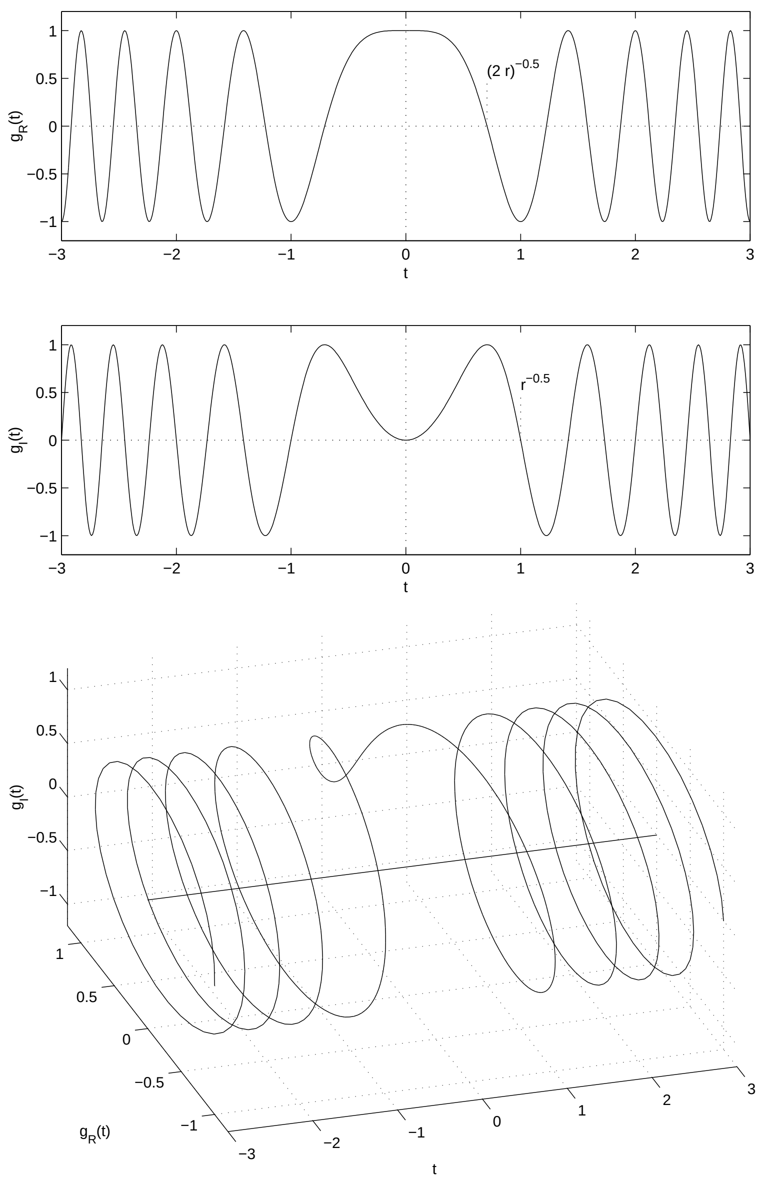

Figure 2.9: Complex FM signal $x_{FM}(t) = \exp(j\pi r t^2)$

2.4.1 Linear FM signal

The complex signal $\exp(j\pi r t^2)$, where r is a constant has some interesting properties. From the expression it can be seen that the amplitude of the signal is 1, and that the phase of the signal is given by:

$$\theta(t) = \pi r t^2. \tag{2.65}$$

The signal is illustrated in Fig. 2.9. The spectrum of the signal is formally given by:

$$X_{FM}(f) \int_{-\infty}^{\infty} e^{j\pi r t^2} e^{-j2\pi f t} dt = e^{-j\pi f^2/r} \int_{-\infty}^{\infty} e^{j\pi r(t-f/r)t^2} dt = \sqrt{\frac{j}{r}} e^{-j\pi f^2/r}. \tag{2.66}$$

The amplitude spectrum of the signal is constant (frequency independent), and its phase spectrum varies quadratically with frequency. Notice the similarity between the signal and its spectrum.

The instantaneous frequency f_i is found to be:

$$f_i = \frac{\theta'}{2\pi} = rt. \tag{2.67}$$

The instantaneous frequency f_i increases linearly with time at rate defined by the constant r (measured for example in Hz/sec.). This is the reason why this signal is called linear frequency modulated signal/pulse. Figure 2.9 shows the signal as a function of time.

It can be seen, that this signal does not belong to the set of signals with finite energy. However, the signal

$$x(t) = a(t)exp(j\pi rt^2), \tag{2.68}$$

where $a(t)$ is appropriately chosen real signal, does. The spectrum $X(f)$ of this signal can be found from:

$$X(f) = e^{-j\pi f^2/r} \int_{-\infty}^{\infty} a(t) e^{j\pi r(f/r-t)^2} dt. \tag{2.69}$$

In other words, the shape of the amplitude spectrum $|X(f)|$ is determined by the convolution of two functions $A(f)$ and $\exp(j\pi r f^2)$. While the result of this convolution cannot always be calculated directly, one can for "very large" values of the parameter r use an approximation, which is described in the following paragraphs.

Consider the following signal:

$$\Delta_p(t) = \sqrt{\frac{p}{j}} \exp(j\pi p t^2). \tag{2.70}$$

It can be shown that $\Delta_p(t) \to \delta(t)$, when $p \to \infty$. Correspondingly one can show, that for large values of the product rT, where T is the "duration" of $a(t)$, we have:

$$X(f) \approx a\left(\frac{f}{r}\right) \cdot \sqrt{\frac{j}{r}} \cdot e^{-j\pi f^2/r} \tag{2.71}$$

if $a(f)$ is continuous. Under these conditions, $|X(f)|$ has almost the same shape os $|a(t)|$ (appropriately scaled though).

Consider the Fourier transform pair:

$$a(t) * \exp(j\pi r t^2) \leftrightarrow X_a(f) \cdot \sqrt{\frac{j}{r}} \exp(-j\pi f^2/r), \tag{2.72}$$

where $a(t) \leftrightarrow X_a(f)$. When the product rT is small, one can use similar considerations as above and show that the signal $a(t) * \exp(j\pi r t^2)$ has an envelope, whose shape is given by $|X_a(f)|$. $|X_a(f)|$ must be appropriately delayed and scaled.

Figure 2.10 shows an example of the result from the convolution operation for a rectangular signal, whose duration is varied.

Notice that if $a(t)$ is time-limited, i.e. starts at $t = t_1$ and ends at $t = t_2$, then the a FM-signal with a "start" and "end" frequencies rt_1 and rt_2 is created. This can also be achieved by applying the frequency-shift property of the Fourier transform.

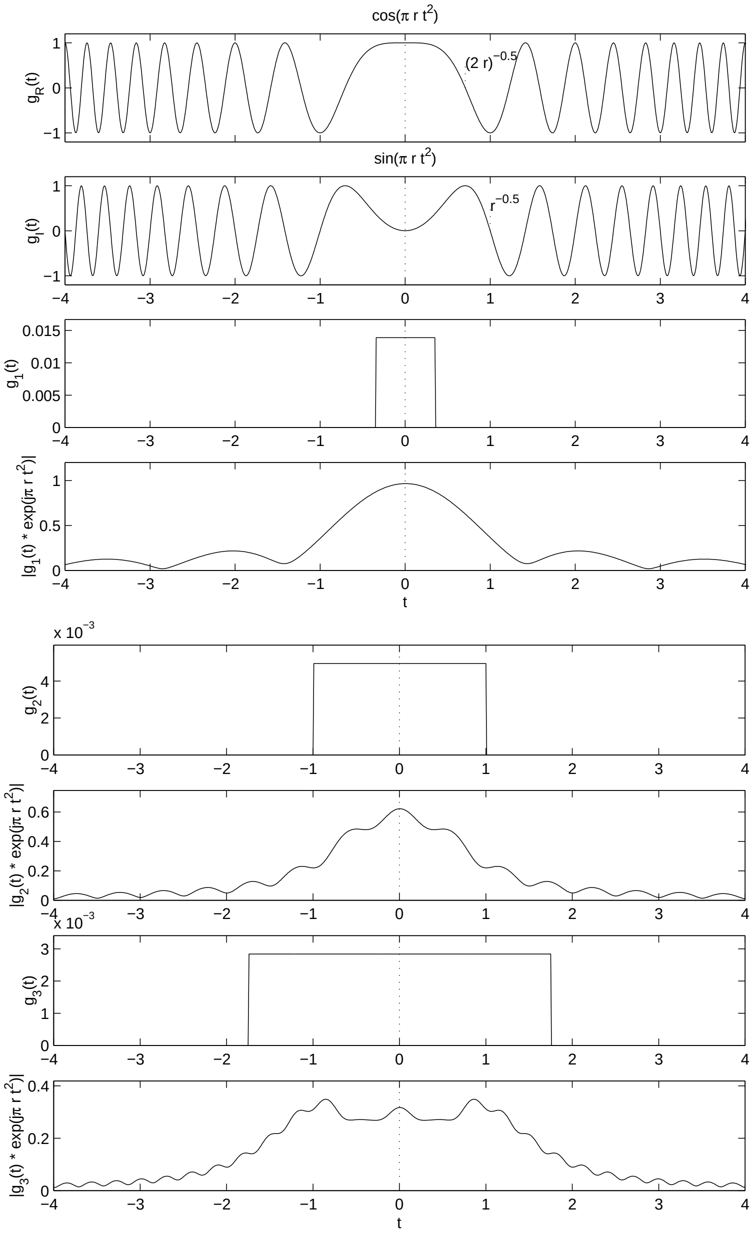

Figure 2.10: Averaged complex FM signals

2.5 Example problems

Example 1
Plot the magnitude of the transfer function of a high-pass filter, which is described by:

$$y(n) = x(n) - x(n-1) \tag{2.73}$$

Solution
Using the rules of linearity and time shift we obtain:

$$\begin{aligned} Y(f) &= X(f)\underbrace{(1 - e^{-j2\pi f})}_{H(f)} \\ |H(f)| &= |1 - e^{-j2\pi f}| \\ &= 2\underbrace{|e^{-j\pi f}|}_{=1}|\sin(\pi f)| \end{aligned}$$

The spectrum is illustrated in Fig. 2.11

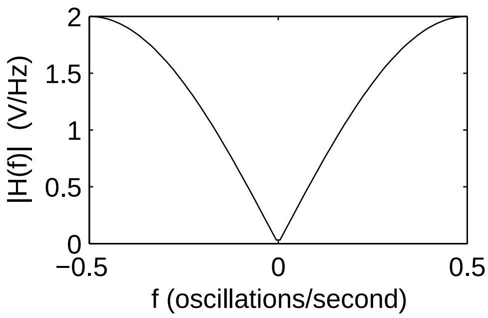

Figure 2.11: The transfer function of the system from Example 1.